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Buckling or Transverse Deflections of Unsymmetrically Laminated Plates Subjected to In-Plane Loads

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Numerous research publications purport to give in-plane buckling loads for unsymmetrically laminated composite plates. In many of these cases, bifurcation buckling is impossible; that is, transverse deflection is initiated, regardless of the magnitude of the loading. In the present work, finite element studies are made of four types of unsymmetrical laminates, including antisymmetrical ones. Twelve sets of edge constraints are considered. Maximum values of transverse deflection are determined for uniform and linearly varying in-plane boundary forces, including uniaxial, biaxial, and shear. Cases where the plate remains flat are thereby also identified. Results confirm a theoretical analysis previously made. Cases for which improper buckling results are often reported in the published literature are identified.

Introduction

IT is generally known that laminated composite flat plates that are fabricated in stacking sequences that are unsymmetric with respect to their midplanes will typically bend and twist when subjected to in-plane compressive loads. This is due to the coupling between in-plane and out-of-plane displacements that exists in the stiffness of such plates.

For certain special cases of stacking sequences, boundary conditions and in-plane loading, unsymmetric laminates will not bend or twist when compressive loads are applied to their midplanes but will remain flat until the bifurcation buckling load is reached. Theoretical values of the buckling load in such cases may be obtained by solving an eigenvalue problem. Examples of problems having buckling loads were shown by Whitney and Leissa[1,2] for uniaxially and biaxially loaded, angle-ply rectangular plates having two types of simply supported edge conditions. For one type, where the edges are subjected to normal in-plane constraint ($u_n = 0$), closed-form, exact solutions for the eigenvalues (nondimensional buckling loads) were obtained.[1] For the other type, where no normal in-plane constraint is applied, eigenvalues were obtained by accurate, but approximate, methods.[2]

In the past two decades many analyses have been published for the buckling of unsymmetric laminates where such buckling cannot physically exist,[3-22] although in a few instances[4,10,11] the researchers did realize that transverse deflection did occur in the prebuckled state and that bifurcation buckling did not take place. One configuration often used in such incorrect analyses is the simply supported, antisymmetrically laminated cross-ply plate, having uniaxial or biaxial loading. Others often used are angle-ply or cross-ply plates having the uniaxially loaded edges simply supported or clamped, with the other two edges free.

In 1986 the second author published a paper[23] that proved under what conditions laminated plates remain flat under in-plane loadings; that is, for what types of stacking sequences, boundary conditions, and in-plane loadings buckling situations may arise. It was proved that buckling may always occur for symmetric laminates with arbitrary in-plane loading and boundary conditions. For unsymmetric laminates it was proved that buckling may occur under uniform and/or linearly varying in-plane loading (Fig. 1) in the following situations: 1) general unsymmetric laminates, provided that proper bending and twisting moments are applied at the edges and that additional transverse edge forces are applied in the case of linearly varying loads; 2) antisymmetric angle plies, with proper edge twisting moments for uniaxial or biaxial loading, and edge bending moments for in-plane shear loading; and 3) antisymmetric cross plies, with proper edge bending moments for uniaxial or biaxial loading, but no edge moments required for in-plane shear loading.

It was also proved[23] that, when in-plane loadings that are neither constant nor linearly varying are applied to any unsymmetrical laminate, the plate will not remain flat, in general, unless proper transverse pressure and/or tangential tractions are simultaneously applied to the plate surfaces, which is a practically unrealistic situation.

The incorrect analyses[3-22] mentioned earlier that yield physically incorrect solutions arise because mathematically homogeneous differential equations (or corresponding energy principles) governing the neutral equilibrium state present in bifurcation buckling, along with homogeneous boundary conditions, are applied that are inconsistent with the prebuckling boundary conditions. More will be said about this later. Such a trap is easy to fall into. Indeed, the second author himself fell into it (*after* writing Ref. 23!) in a study of the accuracy of the reduced stiffness theoretical model for representing unsymmetrically laminated plates.[24] There, results for static transverse loading, free vibrations, and buckling (uniaxial and

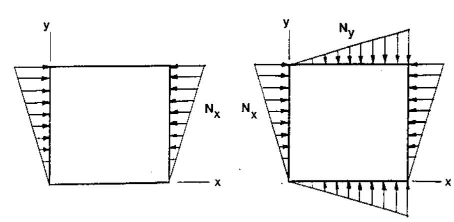

Fig. 1 Uniaxial and biaxial linearly varying in-plane stress resultants.

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biaxial loading) for cross-ply and angle-ply laminates having simply supported edges were presented, although the cross-ply laminates do not remain flat under uniaxial or biaxial loadings.

The present paper is being written with the following objectives in mind: 1) to substantiate the conclusions reached in the previous work[23] by means of a finite element analysis capable obtaining correct and accurate results; 2) to elaborate on these conclusions and show, in detail, for what types of laminates, boundary conditions, and in-plane loadings bifurcation buckling may occur; and 3) to identify further what particular problems have misled analysts in the past, so that researchers and designers may avoid these pitfalls in the future. The finite element method is used because exact solutions for the transverse displacements due to in-plane loadings are possible in only certain very special cases.

A byproduct of this work is a set of extensive results for the small, transverse deflections of unsymmetrically laminated plates having various types of edge conditions, subjected to several types of in-plane loading. Although a few results are available in the literature for nonlinear, large transverse deflections,[25-27] the authors are unaware of any previous studies of the linear, small transverse deflections of unsymmetric laminates having a variety of in-plane loading and edge conditions.

Description of the Finite Element Analysis

The strain- and curvature-displacement relations for plates are[28]

$$\epsilon_x^0 = \frac{\partial u_0}{\partial x}, \qquad \epsilon_y^0 = \frac{\partial v_0}{\partial y}, \qquad \gamma_{xy}^0 = \frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial y}$$
$$\kappa_x = -\frac{\partial^2 w_0}{\partial x^2}, \qquad \kappa_y = -\frac{\partial^2 w_0}{\partial y^2}, \qquad \tau = -2\frac{\partial^2 w_0}{\partial x \partial y} \tag{1}$$

where ϵ_x^0 and ϵ_y^0 are normal strains and γ_{xy}^0 is the shear strain; κ_x and κ_y are bending curvatures, and τ is the twist; and u_0 and v_0 are in-plane displacements, and w_0 is the transverse displacement. All of these quantities are measured at the midplane of the plate.

Strains at arbitrary points throughout the thickness of the plate follow from the Kirchhoff hypothesis (normals to the midplane remain straight and normal during deformation of the plate):

$$\epsilon_x = \epsilon_x^0 + \zeta\kappa_x, \qquad \epsilon_y = \epsilon_y^0 + \zeta\kappa_y, \qquad \gamma_{xy} = \gamma_{xy}^0 + \zeta\tau \tag{2}$$

where ζ is a coordinate measured normal to the midplane.

The force and moment resultants are written in terms of the midplane strains and curvature changes as[28]

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \tau \end{bmatrix} \tag{3}$$

where N_x and N_y are in-plane normal force (resultants), and N_{xy} is the shear stress; M_x and M_y are in-plane bending moment resultants, and M_{xy} is the twisting moment, all per unit length. The A_{ij}, D_{ij}, and B_{ij} are the stretching, bending, and stretching-bending (coupling) stiffness coefficients. For symmetrically laminated plates all B_{ij} are zero, but for unsymmetrical laminates they are not.

The total potential energy of a plate element loaded with p_x, p_y, and p_n pressures, in the x, y, and z directions, respectively, as well as edge loads is

$$\Pi = \frac{1}{2}\int_A\int\Big(N_x\epsilon_x^0 + N_y\epsilon_y^0 + N_{xy}\gamma_{xy}^0 + M_x\kappa_x + M_y\kappa_y + M_{xy}\tau - p_x u - p_y v - p_n w\Big)\,dA - \int_L\Big(-M_n\frac{\partial w}{\partial n} + V_n w + N_n u_n + N_{nt}u_t\Big)\,dL \tag{4}$$

where A and L are the element domain and boundary, respectively, and n and t represent the directions normal and tangential to the boundary, respectively. The (^{0}) in the previous equations will be dropped in the subsequent analysis for convenience.

A typical quadrilateral plate finite element is used. The displacements u, v, and w are interpolated within the plate element using the following expressions:

$$u = \sum_{j=1}^{J_1} u_j\Psi_j(x,y) \qquad v = \sum_{j=1}^{J_2} v_j\Psi_j(x,y)$$
$$w = \sum_{j=1}^{J_3} w_j\Psi_j(x,y) \tag{5}$$

where Ψ_j represent the shape functions, and u_j, v_j, and w_j are nodal point displacements.

Substituting Eqs. (5) into the strain- and curvature-displacement relations (1) yields

$$\epsilon = \boldsymbol{B}\boldsymbol{u} \tag{6}$$

where ϵ is the vector of midplane strains and curvature changes (Eq. 3), and $\boldsymbol{u}$ is the vector of nodal displacements.

Taking the variation of the potential energy (4) yields

$$\delta\Pi = \frac{1}{2}\int_A\int\Big(N_x\ N_y\ N_{xy}\ M_x\ M_y\ M_{xy}\Big)\Big(\delta\epsilon_x\ \delta\epsilon_y\ \delta\gamma_{xy}\ \delta\kappa_x\ \delta\kappa_y\ \delta\tau\Big)^T\,dA - \int_A\int p_x\delta u + p_y\delta v + p_n\delta w\,dA - \int_L\Big(-M_n\delta\frac{\partial w}{\partial n} + V_n\delta w + N_n\delta u_n + N_{nt}\delta u_t\Big)\,dL \tag{7}$$

Substituting Eqs. (3) and (6) into Eq. (7) yields the following equation:

$$\boldsymbol{K}\boldsymbol{u} = \boldsymbol{f} \tag{8}$$

where $\boldsymbol{K}$ and $\boldsymbol{f}$ are the plate element stiffness matrix and generalized force vector, respectively.

By specifying the values J_1, and J_2, and J_3, the stiffness matrix and generalized force vectors can be derived for different elements. Algebraic polynomial shape functions and quadrilateral isoparametric elements of four and eight nodes were used.[29]

To verify the correctness of the stiffness matrix and to determine the element convergence characteristics, solutions using the previous elements were compared with exact solutions for laminated plates with shear diaphragm boundaries subjected to uniform transverse loading.[1,8] Table 1 shows a comparison between results obtained for cross-ply laminated plates using various elements and mesh sizes. The plate is square of side length a; h is its thickness; E_1, E_2, and G_{12} are orthotropic elastic moduli for the composite material plies; and ν_{12} is Poisson's ratio. Although the previous elements are nonconforming, they give reasonably accurate results for the unsymmetrically laminated plates. A mesh of 100 eight-noded elements is chosen for the subsequent analysis. Thus, the plate has 1705 discrete degrees of freedom before the boundary conditions are applied.

Table 1 Comparison of parameters obtained by exact and finite element solutions at the center of a uniformly loaded simply supported [0,90] laminated plate[a]

Analysis	$\bar{w}$	$\bar{N}_x$	$\bar{N}_y$	$\bar{N}_{xy}$	$\bar{M}_x$	$\bar{M}_y$	$\bar{M}_{xy}$
Exact	2912	−4	4	0	519	519	0
36 four-noded elements	2662	−2	2	0	528	528	0
100 four-noded elements	2819	−7	7	0	524	524	0
400 four-noded elements	2887	−2	2	0	522	522	0
36 eight-noded elements	2797	−8	8	0	492	492	0
100 eight-noded elements	2819	−5	5	0	514	514	0

[a] $a/h = 100$, $E_1/E_2 = 15.4$, $G_{12}/E_2 = 0.79$, $\nu_{12} = 0.3$, $\bar{w} = 10^4 \times wE_1h^3/qa^4$, $[\bar{M}_x, \bar{M}_y, \bar{M}_{xy}] = 10^4 \times [M_x, M_y, M_{xy}]/qa^2$, $[\bar{N}_x, \bar{N}_y, \bar{N}_{xy}] = 10^2 \times [N_x, N_y, N_{xy}]/qa$

Numerical Results

The finite element computer program described in the preceding section was used to perform a deflection and bending moment analysis for a set of square, laminated composite plates having various layer stacking sequences, boundary conditions, and in-plane loadings. Layers were orthotropic, with material properties representative of a graphite/epoxy composite: $E_1 = 104$ GPa (15×10^6 psi), $E_2 = 6.9$ GPa (10^6 psi), G_{12} GPa (0.5×10^6 psi), and $\nu_{12} = 0.25$. Plate dimensions were taken as 2.54 m $\times$ 2.54 m (100×100 in.), with a total thickness of $h = 25.4$ mm (1 in.).

Four types of stacking sequences were used, two having two layers, and two having four layers, with all layers in a given laminate having the same thickness: 1) [0,90]—antisymmetric cross ply, 2) [30, −30]—antisymmetric angleply, 3) [30, 30, 30, −30]—unsymmetric angle ply, and 4) [0, 15, 30, 45]—unsymmetric general. Elements of the stiffness matrix in Eq. (3) are given in Table 2 for each of the four laminates used. By studying Eq. (3) and the values given in Table 2, especially the B_{ij} elements, one can understand better how an unsymmetric laminate deforms due to in-plane loads.

Twelve combinations of edge conditions were investigated, which fall into three groups, each having four combinations: 1) opposite edges that are either simply supported or clamped; 2) two opposite edges that are simply supported or clamped and the other two that have no transverse displacement constraint; and 3) representative cases that have only one axis of planform symmetry in the edge conditions, compared with two symmetry axes for the first two groups. The 12 combinations are shown in the first column of Tables 3–6. The designation CSCF, for example, identifies a plate that has the edge $x = 0$ clamped, whereas the remaining three edges are simply supported, clamped, and free as one moves continuously around the boundary. The symbol "X" identifies an edge that has a slope constraint ($\partial w/\partial n = 0$) only and is therefore capable of generating reactive, normal edge moment but no transverse shearing force. It may be called a "sliding clamp," with the sliding occurring in the transverse direction.

The aforementioned combinations are studied because of their frequent appearance in the literature for buckling studies. Furthermore, they give an indication of the effect of each added boundary constraint on the transverse displacement. In the cases where the plate remains flat, the preceding analysis can give the values of the moments required on the edges to keep it flat.

Since this is a static problem, either the forces (or moments) or the displacement (or slope) may be specified at each boundary. In the present study, in-plane forces are applied; subsequently no constraints are applied to the in-plane displacements.

Seven types of in-plane loading were applied to each plate, as shown across the top of Tables 3–6. The first four are uniform (constant), whereas the last three are linearly varying. The uniform loading cases include uniaxial loading of unit magnitude in each direction, biaxial [$N_x = N_y = 175$ N/m (1 b/in.)] loading, and shear loading. The stress resultant N_x, for example, is applied to the edges $x = 0$ and 2.54 m (100 in). Data are given only for *positive* in-plane shear loading. In Fig. 2 a [30, −30] antisymmetric angle-ply plate is subjected to positive shear loading (N_{xy}). For negative shear the directions of the edge loadings would all be reversed. Although buckling loads of symmetrically laminated, angle-ply plates are typically different for positive and negative shear,[30] the linear transverse deflections of the present analysis for unsymmetric laminates are found to simply change sign as the shear stress changes sign. The linearly varying loads used in Tables 3–6 are axial and biaxial, but are not shear. The linearly varying N_x and N_y have *average* values of unit load (maximum values of two). Linearly varying shear stress is unacceptable, because it violates the equations of in-plane equilibrium for all plates.

Tables 3–6 list the *maximum* values of transverse deflection determined for a plate with prescribed unit in-plane loading and edge conditions. These data were obtained from deflection contour plots made for each case ($7 \times 12 \times 4 = 336$ cases). In certain cases the maximum deflection occurs at the plate center, but in most cases it does not.

Discussion of Results

Consider first the antisymmetrically laminated cross-ply (ALCP) plate. In the previous theoretical analysis,[23] it was shown that such plates require edge bending moments to be applied for uniaxial or biaxial loading, if the plate is to remain flat. This is confirmed in the first three numerical columns of Table 3 where one sees that zero deflection occurs only when all edges have either rigid (C) or sliding (X) clamps.

Numerous researchers have published uniaxial bifurcation buckling loads for simply supported ALCP plates.[3,7,10,12–15,19–21] The first row of data in Table 3 shows clearly that such buckling cannot occur. Indeed, quite a large deflection (116.1×10^{-4} mm) takes place at the plate center due to the uniaxial compressive loading; that is, it is two

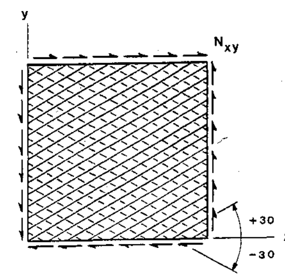

Fig. 2 Two-layer angle-ply (±30 deg) plate subjected to positive in-plane shear stresses.

Table 2 Nondimensional stiffness coefficients for the four laminates used[a]

Coefficient	Laminate			
	[0,90]	[30, −30]	[30,30,30, −30]	[0,15,30,45]
$\bar{A}_{11}$	8.03	9.00	9.00	10.50
$\bar{A}_{22}$	8.03	1.98	1.98	2.18
$\bar{A}_{66}$	0.50	3.04	3.04	2.20
$\bar{A}_{12}$	0.25	2.79	2.79	1.94
$\bar{A}_{16}$	0	0	2.26	2.81
$\bar{A}_{26}$	0	0	0.79	1.34
$\bar{B}_{11}$	−1.76	0	0	−1.11
$\bar{B}_{22}$	1.76	0	0	0.37
$\bar{B}_{66}$	0	0	0	0.37
$\bar{B}_{12}$	0	0	0	0.37
$\bar{B}_{16}$	0	−1.13	−0.85	0.37
$\bar{B}_{26}$	0	−0.39	−0.30	0.37
$\bar{D}_{11}$	6.69	7.50	7.50	8.34
$\bar{D}_{22}$	6.69	1.65	1.65	2.22
$\bar{D}_{66}$	0.42	2.54	2.54	1.83
$\bar{D}_{12}$	0.21	2.33	2.33	1.62
$\bar{D}_{16}$	0	0	0.47	1.68
$\bar{D}_{26}$	0	0	0.16	1.38

orders of magnitude larger than when the loaded edges are clamped (1.3×10^{-4} mm), and three times as large as when the unloaded edges are clamped (39.9×10^{-4} mm). Another recent study[31] made theoretical and experimental investigations of uniaxially loaded ALCP plates having CSCS edges. Theoretical bifurcation buckling loads were obtained by two methods—Ritz and finite elements—although it was pointed out that all experimental data indicated transverse deflection from the onset of the loading (instead of buckling).

Extensive buckling loads are also found in the literature for ALCP plates having their loaded edges either simply supported or clamped and the unloaded edges free.[10,11,15,17,18,21] Table 3 shows a very large deflection for the SFSF case (244.3×10^{-4} mm), and a small one for the CFCF case (-2.5×10^{-4} mm). Thus, for the CFCF case, one could more likely identify *apparent* buckling in experiments.[11] Uniaxial buckling loads were also claimed[22] for such plates with SESF edges, where "E" denotes a simply supported edge with elastic rotational constraint, but Table 3 shows that buckling cannot occur in this case either because the SESF case is more flexible than the SXSX, which has relatively large deflection.

It was previously shown[23] that ALCP plates loaded by in-plane shear (N_{xy}) require no bending or twisting moments along their edges to remain flat. This is substantiated for all edge conditions in Table 3.

For linearly varying N_x and N_y it was proved that linearly varying edge moments and uniform transverse shears must be applied at all edges to keep an ALCP plate flat.[23] This condition is possible only for CCCC edge conditions. Table 3 shows very small maximum deflections ($\pm0.3\times10^{-4}$ mm) occurring for such edge conditions, which are the result of numerical inaccuracies in the finite element computations. Small deflections, which are not due to numerical inaccuracy, are observed for CXCX and CCCX edges in Table 3. Thus, the missing reactive shear forces on one or two edges are not as important as the reactive moments in providing transverse stiffness (compare CXCX with CSCS results).

The conclusions reached earlier for the ALCP plate are all valid also for general, *unsymmetrically* laminated cross plies.

Results for the antisymmetrically laminated angle-ply (ALAP) plate are shown in Table 4. According to theoretical analysis,[23] such plates require edge twisting moments for uniaxial or biaxial loading if they are to remain flat. The twisting moments may be supplied by either clamped (C) or simply supported (S) edges, but not by sliding clamps (X). That conclusion is supported by the data in the first three columns of Table 4 for SSSS, CSCS, SCSC, and CCCC plates. However, it is noted that only small values of deflection arise for the other eight types of edge conditions, where twisting moment is unavailable at one or two of the edges. That these deflections are small could explain why researchers have claimed experimental buckling loads for uniaxially loaded ALAP plates with CFCF[10] or SFSF[4] plates. However, there is no justification for the *theoretical* bifurcation buckling loads obtained from eigenvalue problems for CFCF,[11] SFSF,[9,10,17,18] and SCSF[9,16,18] plates.

Table 3 Maximum deflection ($w_{max}\times10^4$ mm) of a square, antisymmetric, cross-ply [0,90] plate subjected to in-plane loads

Edge conditions	Uniform N_x	Uniform N_y	Uniform $N_x=N_y$	Uniform N_{xy}	Linearly varying N_x	Linearly varying N_y	Linearly varying $N_x=N_y$
SSSS	$\underline{116.1}$[a]	−116.1	±13.5	0	116.3	−116.3	±22.6
CSCS	1.3	−39.9	−38.6	0	2.0	−39.6	−39.1
SCSC	39.9	−1.3	38.6	0	39.6	−2.0	39.1
CCCC	0	0	0	0	−0.3	0.3	±0.3
SFSF	$\underline{244.3}$	112.8	357.1	0	422.4	118.4	534.9
CFCF	−2.5	$\underline{83.8}$	$\underline{81.3}$	0	−3.8	86.9	85.1
SXSX	$\underline{244.3}$	$\underline{-7.6}$	$\underline{236.7}$	0	340.9	−7.9	333.2
CXCX	0	0	0	0	1.3	0.5	1.5
SSSF	268.7	124.0	392.9	0	432.1	129.3	556.0
CSCF	−2.8	88.9	86.1	0	−3.8	91.7	87.9
SCSX	200.7	−62.7	194.6	0	310.6	−6.6	304.3
CCCX	0	0	0	0	1.0	0.5	1.5

[a]Underlined values are when maximum deflection is at center of plate ($x=y=1.27$ m).

Table 4 Maximum deflection ($w_{max}\times10^4$ mm) of a square, antisymmetric, angle-ply [30, −30] plate subjected to inplane loads

Edge conditions	Uniform N_x	Uniform N_y	Uniform $N_x=N_y$	Uniform N_{xy}	Linearly varying N_x	Linearly varying N_y	Linearly varying $N_x=N_y$
SSSS	0	0	0	$\underline{-177.8}$[a]	±0.5	±1.3	±1.5
CSCS	0	0	0	−20.3	±0.5	±1.0	±1.0
SCSC	0	0	0	$\underline{-83.8}$	±0.5	±1.0	±1.3
CCCC	0	0	0	0	±0.3	±0.8	±1.0
SFSF	±0.5	±0.3	±1.0	−351.0	±1.5	±40.6	±41.1
CFCF	±0.0	±0.8	±0.8	52.6	±0.8	±9.4	±9.4
SXSX	±0.3	±0.3	±0.5	−354.1	±0.8	±21.6	±21.8
CXCX	±0.0	±0.5	±0.5	0	±0.5	±5.1	±5.1
SSSF	±0.5	±0.3	±1.0	−284.5	±1.5	47.8	48.0
CSCF	±0.0	±0.8	±0.8	52.3	±0.8	9.7	9.4
SCSX	±0.3	±0.0	±0.5	−259.3	±0.8	27.4	27.7
CCCX	±0.0	±0.5	±0.5	0	±0.5	5.1	5.1

[a]Underlined values are when maximum deflection is at center of plate ($x=y=1.27$ m).

It was previously proved[23] that ALAP plates loaded by in-plane shear (N_{xy}) must have edge bending moments for buckling to occur. This is clearly seen in Table 4, where rigid or sliding clamps (C or X) are required along each edge to have no transverse deflection. It is seen that quite large deflections occur for SSSS plates loaded in shear. Recently published buckling results for SSSS ALAP plates loaded in shear[12] are therefore improper. A buckling optimization procedure which was applied to such plates[5] is also invalid.

The results shown in Table 4 for linearly varying N_x and N_y are not quite consistent with the previous analysis.[23] According to the analysis, ALAP plates should require linearly varying twisting moments and uniform transverse shears along the edges to remain flat. The first four sets of edge conditions (SSSS, CSCS, SCSC, and CCCC) in Table 4 can provide the needed restraints, but small deflections are observed. This is probably because the finite element model can only approximate the linear stress variation along the boundaries by means of forces applied at discrete boundary points. Much larger deflections are seen in Table 4 for the other eight sets of boundary conditions, all of which have at least one edge (F or X) that cannot provide twisting moments or transverse shears.

It should be mentioned that the ALAP plate used for the results of Table 4 has alternation ply angles ($[+\theta, -\theta]$). However, ALAP plates having more general stacking sequences (e.g., $[+\theta_1, +\theta_2, -\theta_2, -\theta_1]$) would yield the same general conclusions (e.g., remaining flat under uniaxial or biaxial in-plane loadings). For all ALAP plates, $B_{16}=B_{26}\neq0$, but all other B_{ij} are zero, which is the common criterion determining whether the plates remain flat.

The unsymmetrically laminated, four-layer plate of Table 5 may be considered the same as an antisymmetrically laminated [30,30, −30, −30] one (which is the same as the [30, −30] plate of Table 4), except that one of the inner layers has been changed from −30 to +30 deg. Both bending and twisting moments must be supplied to unsymmetric laminates for them to remain flat, regardless of the in-plane loading. Therefore, the maximum transverse deflections shown in Table 5 for CCCC edges are all nearly zero. An interesting anomaly is that, for uniform in-plane loads, the sliding clamp (X) causes no significant decrease in plate stiffness from that of the rigid clamp (C), provided that at least two of the edges are rigidly clamped (compare CCCC, CXCX, and CCCX data in Table 5). Comparing Tables 4 and 5 further, one sees that typically the maximum transverse deflection occurring for the [30, 30,

Table 5 Maximum deflection ($w_{max} \times 10^4$ mm) of a square, unsymmetric, angle-ply [30,30,30, −30] plate subjected to in-plane loads

Edge	Uniform				Linearly varying		
conditions	N_x	N_y	$N_x = N_y$	N_{xy}	N_x	N_y	$N_x = N_y$
SSSS	−28.4[a]	−9.9	−38.4	−125.7	−30.2	−9.4	−38.9
CSCS	−3.0	−1.0	−4.1	−14.5	−6.6	−1.5	−7.9
SCSC	−14.0	−4.8	−18.8	−62.7	−13.2	−5.8	−18.8
CCCC	0.5	0.3	0.8	0.5	1.8	0.8	1.5
SFSF	−53.6	−18.8	−72.1	−234.7	−76.5	−62.7	−97.0
CFCF	9.9	3.3	13.2	41.4	15.0	9.9	24.9
SXSX	−54.4	−19.1	−73.4	−237.5	−82.3	−39.4	−82.0
CXCX	0.5	0.3	0.8	0.5	−3.8	4.3	1.8
SSSF	−45.0	−15.7	−60.5	−196.9	−71.4	24.9	−59.2
CSCF	9.9	3.3	13.2	41.4	15.0	10.2	24.9
SCSX	−41.9	−14.7	−56.6	−183.4	−74.4	8.4	−66.3
CCCX	0.5	0.3	0.8	0.5	−3.8	4.3	1.5

[a]Underlined values are when maximum deflection is at center of plate ($x = y = 1.27$ m).

Table 6 Maximum deflection ($w_{max} \times 10^4$ mm) of a square, unsymmetric, general [0,15,30,45] plate subjected to in-plane loads

Edge	Uniform				Linearly varying		
conditions	N_x	N_y	$N_x = N_y$	N_{xy}	N_x	N_y	$N_x = N_y$
SSSS	99.3[a]	−96.8	−8.1	196.9	101.9	−98.8	27.1
CSCS	2.8	−16.8	−14.0	19.6	6.1	−15.7	−14.2
SCSC	53.3	−33.3	19.8	87.4	52.8	−41.1	30.5
CCCC	0	−0.5	−0.5	0.8	−1.3	2.5	1.8
SFSF	206.8	−127.2	142.5	322.8	317.2	−170.2	161.5
CFCF	−7.1	41.1	34.0	−47.2	−15.2	77.0	73.2
SXSX	196.3	−121.2	75.9	317.5	279.9	−180.3	111.5
CXCX	0.0	0.8	0.8	0.8	−2.5	24.4	21.8
SSSF	194.1	−114.0	133.6	283.0	309.9	−171.5	153.7
CSCF	−7.1	41.7	34.5	−48.3	−15.2	16.8	−14.2
SCSX	162.1	−99.6	62.7	261.9	257.0	−167.4	102.6
CCCX	0.0	0.8	0.8	0.8	−1.3	−18.8	−18.3

[a]Underlined values are when maximum deflection is at center of plate ($x = y = 1.27$ m).

30, −30] plate is considerably more than for the [30, −30] plate, except in the case of shear loading.

The preceding comments for the unsymmetric, angle-ply plate of Table 5 also apply to those of the unsymmetric, general laminate of Table 6. In addition, one observes that the more general [0,15,30,45] stacking sequence increases further the bending-stretching coupling effects causing, typically, much larger transverse deflections.

Finally, it must be realized that *maximum* deflection is not a rigorous measure of the stiffness of a configuration. It is only an approximate indicator. An example of this shortcoming is seen in Table 5. Clearly, the SXSX plate is stiffer than the SFSF plate. However, for all loadings but two, the SXSX maximum transverse deflection data are slightly larger. This is because those larger values occur in highly localized regions, whereas the slightly smaller SFSF deflections occur over larger regions.

Summary and Conclusions

An extensive finite element study has been made of four types of antisymmetrically and unsymmetrically laminated composite plates. Each plate was square and was constrained by 12 combinations of edge conditions, chosen so as to determine clearly the effects of edge restraints. Seven types of in-plane loading were applied to each of these $4 \times 12 = 48$ models. Finite element representations used a sufficiently large number of degrees of freedom (1705) to obtain accurate transverse deflection data.

The results of the numerical study just described were twofold: 1) it verified of the theoretical conclusions of a previous study[23] based on the field equations for unsymmetric laminates, to determine conditions under which bifurcation buckling can exist, and 2) it determined the relative magnitudes of transverse displacement due to in-plane loads, when they do not remain flat, based on linear analysis.

In particular, it was verified that for several cases true buckling (i.e., bifurcation) cannot occur, contrary to claims made otherwise in the published literature, especially, 1) simply supported, antisymmetrically laminated cross-ply (ALCP) plates subjected to uniaxial or biaxial loading; 2) ALCP plates having uniaxially loaded edges simply supported or clamped, and the other edges free or elastically supported; 3) antisymmetrically laminated angle-ply (ALAP) plates having one or more unloaded edges free; and 4) simply supported ALAP plates loaded in shear.

Plates that have *combinations* of the in-plane loadings described earlier (uniform compression, uniform shear, linearly varying compression) will also not remain flat when any one of the loading components by itself causes deflection. Thus, for example, buckling interaction diagrams for combined compression and shear for antisymmetrically laminated, simply supported ALAP or ALCP plates are improper, contrary to published results.[6,12]

It was also shown that generally unsymmetrically laminated plates do not remain flat when subjected to in-plane loads, unless all edges are clamped, and that the magnitude of the maximum transverse deflection for other edge conditions varies greatly with the type of constraints employed.

In relatively few cases will unsymmetrically laminated plates remain flat when subjected to in-plane loads. Consequently, in few cases can true buckling occur. The transverse deflection due to unsymmetric lamination is similar to that which arises in a classical, isotropic, homogeneous plate (or in a column) that is not flat (i.e., a geometric imperfection) or when the loading is applied eccentrically with respect to the midplane.

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